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# Aerodynamic Interference Effects on Oscillating Airfoils with Controls in Ventilated Wind Tunnels

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**Lift interference effects are discussed based on Bland's integral equation. A mathematical existence theory is utilized for which convergence of the numerical method has been proved for general (square-integrable) downwashes. Airloads are computed using orthogonal airfoil polynomial pairs in conjunction with a collocation method which is numerically equivalent to Galerkin's method and complex least squares. Convergence exhibits exponentially decreasing error with the number $n$ of collocation points for smooth downwashes, whereas errors are proportional to $1/n$ for discontinuous downwashes. The latter is reduced to $1/n^{m+1}$ with $m$th order extrapolation to the limit (using $m=2$ we obtain hundredfold error reductions with only a 13% increase of computer time). Numerical results are presented showing acoustic resonance, and the effect of Mach number, ventilation, height to chord ratio, and mode shape on wind tunnel interference. Excellent agreement with experiment is obtained in steady flow, and good agreement is obtained for unsteady flow.**

## I. Introduction

CONTINUED interest in the general problem of high subsonic and transonic aeroelasticity has increased the need for improved methods of aerodynamic analysis and testing. While important advances have been achieved in computational fluid mechanics during the past decade, the essentially nonlinear character of transonic flow results in larger computational expense than do linear theories. Nevertheless, linearized subsonic theories remain as a cornerstone of aerodynamic analysis and, for subcritical Mach numbers, provide an efficient means of performing parametric analyses of various effects of concern. Even so, much remains to be done in the subsonic regime, particularly with regard to unsteady flow in wind tunnels. Whereas wind tunnel interference effects are fairly well understood for steady flow,[1-3] a fully satisfactory theory does not exist for unsteady flow, especially when the walls are ventilated.[1,4-7]

The present paper discusses the problem of predicting unsteady airloads on oscillating airfoil-control systems in subsonic two-dimensional ventilated wind tunnels. It is based on earlier work by Bland[8,9] and Fromme and Golberg,[7,10-12] and extends those results in three directions: 1) the numerical computation of airloads in tunnels with ventilated walls, 2) airfoils with controls, and 3) placing the computational theory on firmer mathematical ground.

Before turning to the problem at hand, we make several observations. First, much of existing wind tunnel interference theory is founded on the incomplete point of view that the effect of boundary conditions at the wall can be determined by knowing only the far flowfield characteristics of the isolated airfoil. While this assumption produces simple and useful engineering approximations to angle-of-attack and Mach number "corrections," it suffers by neglecting the truly coupled nature of interference between the walls and the model. Thus, any rational theory of wind tunnel interference must be based on an appropriate formulation in which this aerodynamic coupling is explicitly present. The analysis used in this paper is based on an integral equation method which correctly accounts for such coupling. Second, whether the governing boundary value problem is solved directly via partial differential equations (PDE) or by reduction to an integral equation is largely a matter of computational method. Whereas integral equations are more difficult to formulate, they enjoy the advantage that the dimension of the space in which the problem must be solved is reduced by one. This undoubtedly accounts for their being a frequent method of choice in subsonic and supersonic flow, and there is now evidence[13] to suggest that in transonic flow as well, nonlinear integral equation methods may be an order of magnitude faster than PDE methods.

## II. Basic Equations

Consider a thin, planar airfoil undergoing simple harmonic motion at frequency $\omega$ rad/s about the center plane of a two-dimensional ventilated wind tunnel (Fig. 1). The walls are considered to extend to infinity upstream and downstream, and the flow is assumed to be inviscid and strictly subsonic, and to be a small disturbance from the freestream flow. Free air conditions are included as an important special case upon removing the walls to infinity.

In general, the displacement amplitude function shown may represent upper, lower, or mean airfoil profiles as well as arbitrary aeroelastic shapes, including rigid body and control surface motions. Displacement is measured positive downward so that positive streamwise derivatives correspond to positive angles of attack. Lift is measured positive upward, and pitching moment is measured about the quarter chord, positive in the direction of increasing angle of attack (nose up). Following customary practice, lengths are nondimensionalized by the airfoil semichord $b$, pressures by freestream dynamic pressure $\frac{1}{2}\rho_\infty v_\infty^2$ and, with the notable exception of Mach number, velocities by freestream velocity $v_\infty$. Streamwise coordinates are measured positive downstream, having value $-1$ at the leading edge and $+1$ at the trailing edge.

Denote by $w$ the downwash amplitude nondimensionalized by the freestream velocity,

$$w(x) = \left(\frac{d}{dx} + ik\right) h(x) \tag{1}$$

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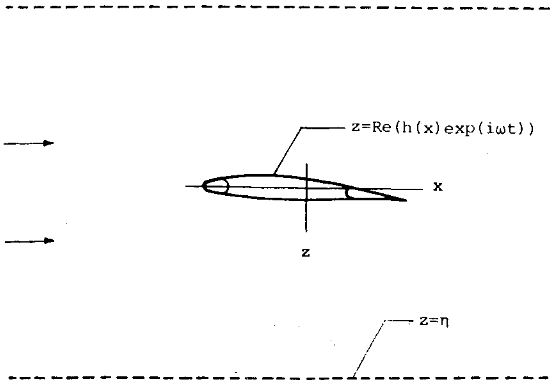


**Fig. 1 Coordinate system and sign convention.**

where $k=\omega b/v_\infty$. Denote by $\phi$ the perturbation velocity potential nondimensionalized by $bv_\infty$ and by $p$ the nondimensional perturbation pressure. Then, $\phi$ satisfies the unsteady wave equation

$$\nabla^2\phi - M^2\left(\frac{\partial}{\partial x}+ik\right)^2\phi=0 \tag{2}$$

where $M=v_\infty/c_\infty$ is the freestream Mach number, and $p$ is given by

$$p=-2\left(\frac{\partial}{\partial x}+ik\right)\phi \tag{3}$$

When the boundary conditions at the upper and lower walls are the same, both pressure and velocity potential can be shown to be antisymmetric functions of $z$; i.e.,

$$p(x,-z)=-p(x,z), \quad \phi(x,-z)=-\phi(x,z) \tag{4}$$

Since the pressure is assumed to be continuous everywhere in the flowfield except across the airfoil where it is permitted to be discontinuous (vortex surface), and since it is assumed to be continuous at the trailing edge (Kutta condition), then

$$p(x,0)=0, \quad |x|>1 \tag{5a}$$

$$p(x,0)=-\tfrac{1}{2}\Delta p(x), \quad |x|<1 \tag{5b}$$

$$p(x,0)=0, \quad x=1 \tag{5c}$$

where $\Delta p$ is the lifting pressure jump across the airfoil. At the surface of the airfoil, the flow is tangent to the surface,

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0}=w(x), \quad |x|<1 \tag{6}$$

and at the wind tunnel walls, we assume that the so-called slotted wall boundary condition

$$p\pm\lambda\frac{\partial p}{\partial z}=0, \quad z=\pm\eta \tag{7}$$

holds. The parameter $\lambda$ is referred to as the ventilation coefficient; when $\lambda=\infty$, Eq. (7) reduces to the closed wall condition; when $\lambda=0$, it reduces to the open jet boundary condition. Davis and Moore[14] first proposed estimating $\lambda$ with a simple formula in terms of slot width, slot spacing and tunnel height. Other investigators[15-20] have discussed various methods for estimating $\lambda$. More recently, Barnwell[6] has concluded that better agreement with experiment is obtained if $\lambda$ is determined by direct correlation with experiment, rather than by using existing theories based on slot geometry, etc.

## III. Bland's Integral Equation

The two-dimensional boundary value problem described by Eqs. (1-7) can be reduced via Fourier transforms to a one-dimensional integral equation

$$w(x)=\frac{\beta}{4\pi}\int_{-1}^{1}A(x-\xi)\Delta p(\xi)\,d\xi \tag{8}$$

relating downwash and pressure jump, where $\beta=\sqrt{1-M^2}$ The kernel $A$ is given by

$$A(x)=\int_{-\infty}^{\infty}e^{isx}\frac{i\sigma}{2(s+k)}\,\frac{\beta\lambda\sigma\sinh\beta\eta\sigma+\cosh\beta\eta\sigma}{\beta\lambda\sigma\cosh\beta\eta\sigma+\sinh\beta\eta\sigma}\,ds \tag{9}$$

where

$$\sigma^2=\left(s+\frac{Mk}{1+M}\right)\left(s-\frac{Mk}{1-M}\right) \tag{10}$$

Technically, Eq. (9) is a distributional inverse Fourier transform[21] and its numerical computation is not straightforward. Following an extensive analysis, Bland[8,9] showed that

$$\begin{aligned}A(x)=&\frac{1}{x}-\frac{ik}{\beta^2}\log|x|\\&+\frac{\pi}{2\beta}(1+\operatorname{sgn}(x))\frac{1+\lambda k\tanh k\eta}{\lambda+(1/k)\tanh k\eta}\exp(-ikx)\\&-\frac{\pi}{\beta\eta}\left[\operatorname{sgn}(x)S'\left(\frac{|x|}{\beta\eta}\right)-\frac{ik\eta}{\beta}S\left(\frac{|x|}{\beta\eta}\right)\right]\exp\frac{ikM^2x}{\beta^2}\\&+\frac{\pi}{2\beta\eta}\left[\operatorname{csch}\frac{\pi x}{2\beta\eta}-\frac{2\beta\eta}{\pi x}+\left(\exp\frac{ikM^2x}{\beta^2}-1\right)\operatorname{csch}\frac{\pi x}{2\beta\eta}\right]\\&-\frac{ik}{\beta^2}\left[\log\left(\frac{1}{x}\tanh\frac{\pi x}{4\beta\eta}\right)-\left(\exp\frac{ikM^2x}{\beta^2}-1\right)\log\tanh\frac{\pi|x|}{4\beta\eta}\right]\end{aligned} \tag{11}$$

where $S$ is given by the infinite series

$$S(\delta)=\sum_{n=1}^{\infty}\left\{\frac{S_n}{\hat\mu_n}\exp(-\hat\mu_n\delta)-\frac{1}{\pi(n-\frac{1}{2})}\exp[-(n-\tfrac{1}{2})\pi\delta]\right\} \tag{12a}$$

$$s_n=1\Big/\left[1+\frac{\gamma}{1+(\gamma\mu_n)^2}\right]\left[1+\left(\frac{k\eta}{\mu_n}\right)^2\right] \tag{12b}$$

$$\hat\mu_n=\mu_n\sqrt{1-\zeta_n^2}, \quad \zeta_n=Mk\eta/\beta\mu_n \tag{12c}$$

and

$$\tan\mu_n+\frac{\lambda}{\eta}\mu_n=0 \tag{12d}$$

Equation (9) is a Fredholm integral equation of the first kind. Its kernel identifies it further as a Cauchy singular equation because of the dominant $1/x$ singularity. The remainder of the kernel consists of a weaker, integrable logarithmic singularity plus an analytic part. It is well known[22] that solutions of such equations are not unique unless the auxiliary Kutta condition is explicitly imposed. Then it follows that solutions for $\Delta p$ possess inverse square root singularities at the leading edge and square root singularities at the trailing edge,

$$\lim_{x\to 1^-}\left|\frac{\Delta p(x)}{\sqrt{1-x}}\right|<\infty, \quad \lim_{x\to -1^+}|\sqrt{1+x}\,\Delta p(x)|<\infty \tag{13}$$

In the case of steady flow in free air, Eq. (11) reduces to the classical vortex kernel

$$A(x) = U(x) = 1/x \tag{14}$$

and Eq. (8) has the closed-form solution given by the Söhngen inversion formula[22-24]

$$\Delta p(x) = \frac{\beta}{4\pi}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{1}{x-\xi}w(\xi)\,\mathrm{d}\xi \tag{15}$$

The preceding considerations of uniqueness, leading- and trailing-edge singularities, and the Söhngen inversion formula led Bland to change the unknown in Eq. (8) from the pressure jump $\Delta p$ to the pressure factor $\psi$ according to

$$\Delta p(x) = \frac{4}{\beta}\sqrt{\frac{1-x}{1+x}}\psi(x) \tag{16}$$

Then Eq. (8) becomes

$$w(x) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}A(x-\xi)\psi(\xi)\,\mathrm{d}\xi \tag{17}$$

The theoretical advantage of Eq. (17) is that it automatically has a unique solution with the correct leading- and trailing-edge singularities. Since this reformulation of the airfoil integral equation into a well-posed problem was first made by Bland,[8,9] we refer to Eq. (17) as Bland's integral equation.

The generalized airfoil kernel $A$ may be split into two parts. Write

$$A(x) = U(x) + K(x) \tag{18}$$

where $K$ is the remaining part, and denote by $\mathcal{A}$, $\mathcal{U}$, and $\mathcal{K}$ the corresponding integral operators ($\mathcal{U}$ is a unitary operator and $\mathcal{K}$ is compact). Then Eq. (17) may be written in operator form as

$$w = \mathcal{A}\psi = (\mathcal{U}+\mathcal{K})\psi \tag{19}$$

and is equivalent to the equation of the second kind

$$\mathcal{U}^{-1}w = \psi + \mathcal{U}^{-1}\mathcal{K}\psi \tag{20}$$

where $\mathcal{U}^{-1}$ is given by Eq. (15). Since equations of the second kind are better understood, both theoretically and computationally, the well-developed methods for equations of the second kind may be applied to the airfoil equation. Further discussion may be found in Ref. 7.

## IV. Fourier Theory of Airfoil Polynomials

From our remarks above and well-known results for the Possio equation,[25] it is reasonable to consider polynomial expansion methods for solving Eq. (17), ones utilizing the airfoil polynomials being of particular importance.

The theory of airfoil polynomials is based on two sets of orthogonal polynomials, one for downwash, the other for pressure. Upon combining Bland's integral equation in terms of the pressure factor with the Söhngen inversion formula for steady flow in free air, there emerges an elegant and completely symmetrical theory between downwash and pressure factor

$$w(x) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}\frac{1}{x-\xi}\psi(\xi)\,\mathrm{d}\xi,$$

$$\psi(x) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{1}{\xi-x}w(\xi)\,\mathrm{d}\xi \tag{21}$$

so much so, in fact, that generalized Fourier coefficients of pressure factor are exactly equal to generalized Fourier coefficients of downwash, i.e.,

$$\langle\psi,\gamma_n\rangle_\gamma = \langle w,\alpha_n\rangle_\alpha \tag{22}$$

To see this, define

$$\alpha_n(x) = \frac{1}{\sqrt{\pi}}\frac{\cos((n-\frac{1}{2})\cos^{-1}x)}{\cos(\frac{1}{2}\cos^{-1}x)},$$

$$\gamma_n(x) = \frac{1}{\sqrt{\pi}}\frac{\sin((n-\frac{1}{2})\cos^{-1}x)}{\sin(\frac{1}{2}\cos^{-1}x)} \tag{23}$$

for $n=1,2,\ldots$ These functions satisfy the following recursion formulas[8,9]

$$\alpha_1(x) = \frac{1}{\sqrt{\pi}},\quad \alpha_2(x) = \frac{1}{\sqrt{\pi}}(-1+2x),$$

$$\alpha_{n+2}(x) = 2x\alpha_{n+1}(x) - \alpha_n(x) \tag{24a}$$

$$\gamma_1(x) = \frac{1}{\sqrt{\pi}},\quad \gamma_2(x) = \frac{1}{\sqrt{\pi}}(1+2x),$$

$$\gamma_{n+2}(x) = 2x\gamma_{n+1}(x) - \gamma_n(x) \tag{24b}$$

which are neutrally stable[10] so that computational errors do not increase with $n$. It is easy to see that $\alpha_n$ and $\gamma_n$ are polynomials of degree $n-1$ with interdigitated zeros. We refer to $\alpha_n$ and $\gamma_n$, respectively, as the $n$th downwash and pressure polynomials. Collectively, they are called airfoil polynomials.

The airfoil polynomial pairs can be shown to represent infinitely many linearly independent solutions to the steady free air problem

$$\mathcal{U}\gamma_n = \alpha_n,\quad n=1,2,\ldots \tag{25}$$

and to be orthonormal with respect to reciprocal downwash and pressure weighting functions; i.e.,

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\alpha_m(x)\alpha_n(x)\,\mathrm{d}x = \int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\gamma_m(x)\gamma_n(x)\,\mathrm{d}x = \delta_{mn} \tag{26}$$

where $\delta_{mn}$ is the Kronecker delta.

Define two complex Hilbert spaces of square integrable functions,

$$\mathcal{L}^2_\alpha = \left\{f:[-1,1]\to\mathbb{C}\;\Big|\;\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,|f(x)|^2\,\mathrm{d}x<\infty\right\} \tag{27a}$$

$$\mathcal{L}^2_\gamma = \left\{f:[-1,1]\to\mathbb{C}\;\Big|\;\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,|f(x)|^2\,\mathrm{d}x<\infty\right\} \tag{27b}$$

called downwash space and pressure space, with respective inner products and norms,

$$\langle f,g\rangle_\alpha = \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}f(x)\bar{g}(x)\,\mathrm{d}x,\quad \|f\|_\alpha = \sqrt{\langle f,f\rangle_\alpha} \tag{27c}$$

$$\langle f,g\rangle_\gamma = \int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)\bar{g}(x)\,\mathrm{d}x,\quad \|f\|_\gamma = \sqrt{\langle f,f\rangle_\gamma} \tag{27d}$$

In this setting, it is easy to show that the sets of functions $\{\alpha_n\}_1^\infty$ and $\{\gamma_n\}_1^\infty$ are complete orthonormal bases for $\mathcal{L}^2_\alpha$ and $\mathcal{L}^2_\gamma$, respectively.[26] Therefore, if

$$w\in\mathcal{L}^2_\alpha \text{ then } w = \sum_{n=1}^{\infty}\langle w,\alpha_n\rangle_\alpha\alpha_n \tag{28a}$$

and if

$$\psi\epsilon\mathcal{L}^2_\alpha \text{ then } \psi=\sum_{n=1}^{\infty}\langle\psi,\gamma_n\rangle_\gamma\gamma_n \tag{28b}$$

Furthermore,

$$\text{if } w\epsilon\mathcal{L}^2_\alpha \text{ and } \mathcal{K}\psi=w, \text{ then } \psi\epsilon\mathcal{L}^2_\gamma \text{ and } \langle\psi,\gamma_n\rangle_\gamma=\langle w,\alpha_n\rangle_\alpha \tag{29}$$

as asserted earlier in Eq. (22), and conversely (see Ref. 25 for historical comments).

Once the Fourier coefficients of the pressure factor have been determined, the pressure jump may be computed from

$$\Delta p(x)=\frac{4}{\beta}\sqrt{\frac{1-x}{1+x}}\sum_{n=1}^{\infty}\langle\psi,\gamma_n\rangle_\gamma\gamma_n(x) \tag{30}$$

Defining section coefficients and the aerodynamic work matrix in the usual way,[27]

$$C_L=\frac{1}{2}\int_{-1}^{1}\Delta p(x)\mathrm{d}x,\quad C_m=-\frac{1}{4}\int_{-1}^{1}\left(x+\frac{1}{2}\right)\Delta p(x)\mathrm{d}x \tag{31}$$

$$w_{mn}=\frac{1}{2}\int_{-1}^{1}h_m(x)\Delta p_n(x)\mathrm{d}x \tag{32}$$

where $\Delta p_n$ is the pressure jump due to displacement mode $h_n$, it then follows[25] easily that

$$C_L=\frac{2\sqrt{\pi}}{\beta}\langle\psi,\gamma_1\rangle_\gamma,\quad C_m=-\frac{\sqrt{\pi}}{2\beta}\langle\psi,\gamma_2\rangle_\gamma \tag{33}$$

$$[w_{mn}]=\frac{2}{\beta}[\langle h_m,\gamma_n\rangle_\gamma][\langle\psi_m,\gamma_n\rangle_\gamma]^* \tag{34}$$

where $\psi_m$ is the pressure factor corresponding to $h_m$ and where * denotes complex conjugate transpose. These particularly simple formulas hold in general, so long as airfoil polynomials are used.

## V. Solution of Bland's Integral Equation

Suppose $w\epsilon\mathcal{L}^2_\alpha$ and assume for the moment that $\psi\epsilon\mathcal{L}^2_\gamma$. Since the pressure polynomials $\{\gamma_n\}_1^\infty$ are a complete orthonormal basis for $\mathcal{L}^2_\gamma$, we seek an approximate solution to Eq. (17) of the form

$$\psi^N=\sum_{n=1}^{N}a_n^N\gamma_n \tag{35}$$

where $N$ is fixed and $a_n^N$ are constants to be determined. Substituting Eq. (35) into Eq. (17) gives a residual downwash error

$$r^N=w-\mathcal{A}\psi^N=\mathcal{A}(\psi-\psi^N) \tag{36}$$

In general, $r^N$ will not vanish everywhere, so various approximation techniques such as collocation, complex least squares, and Galerkin's method may be employed to minimize it in some specific sense. For example, the general collocation equations, introduced into unsteady aerodynamics by Possio,[28] are

$$r^N(x_m)=0,\quad m=1,2,\ldots,N \tag{37}$$

where $x_1,\ldots,x_N$ are distinct downwash collocation nodes. This gives an $N\times N$ matrix equation

$$[C_{mn}^N]\{a_n^N\}=\{w(x_m)\} \tag{38}$$

to be solved for the $a_n^N$, where

$$C_{mn}^N=(\mathcal{A}\gamma_n)(x_m) \tag{39}$$

is the collocation matrix. In the present method, the unitary part $\mathcal{U}\psi_n$ of $\mathcal{A}\psi_n$ is evaluated exactly by the Akheizer-Bland transform [Eq. (26)], the compact part $\mathcal{K}\gamma_n$ is computed by Gaussian quadrature,[25] and the collocation nodes are chosen as the zeros of the downwash polynomial $\alpha_{N+1}$, which gives

$$C_{mn}^N=\alpha_n(x_m^{N+1})+(\mathcal{K}\gamma_n)(x_m^{N+1}) \tag{40}$$

Bland[9] observed that Eq. (40) produced rapid convergence with $N$ for smooth downwashes in closed wall tunnels but offered no proof of convergence nor conditions under which convergence might occur. We observed similar behavior for ventilated tunnels and established a mathematical convergence proof based on a three-way equivalence between collocation, complex least squares, and Galerkin's method.[10] To within quadrature and machine errors, this equivalence is exact, and holds whenever the downwash collocation nodes coincide with the quadrature nodes used for least squares and Galerkin's method. In particular, it has been shown that if $w\epsilon\mathcal{L}^2_\alpha$, then $\psi\epsilon\mathcal{L}^2_\gamma$ and solving Eq. (38) using Eq. (40) gives for each $N$ a solution vector $\{a_n^N\}$ such that

$$\lim_{N\to\infty}a_n^N=\langle\psi,\gamma_n\rangle_\gamma \tag{41}$$

for $n=1,2,\ldots$ This guarantees a computational solution to the general case and, as we shall see, is computationally efficient as well. More recently, tighter error bounds and a direct proof of Eq. (41) based on collocation alone have been given,[11] along with a priori estimates of the relative performance of iteration, singularity subtraction, and the use of reverse flow theorems.[25]

## VI. Convergence of the Collocation Solution

### A. Smooth Downwash

We now present several examples of numerical convergence produced by the computer program TWODI[25] used to solve the equations described earlier. Consider first the special case of unsteady incompressible flow in free air. Select displacement functions $\gamma_n$, i.e., $h_n=\gamma_n$. Note that $\gamma_1$ corresponds to a plunging mode, $\gamma_2$ to pitching about the quarter chord, and $\gamma_3$ to an elastic mode which might be called first bending. Using the exact closed-form Küssner-Schwarz solution[29] as a basis of comparison (see Ref. 30 for formulas), Table 1 and Fig. 2a show the computational convergence with $N$ for lift coefficient in the first three modes. The convergence appears to be generally exponential with $N$. As should be expected, convergence is slightly delayed (although not slowed!) with the higher modes, thereby indicating that a few more terms should be used to maintain accuracy for higher aeroelastic displacements.

As a second example, consider the problem of computing unsteady airloads for an airfoil undergoing pitching oscillations about the 42.5% chord at $M=0.85$ in a wind tunnel with height to chord ratio of 7.5.[9] Figure 2b shows the convergence of $C_{L_\alpha}$ with $N$ for open jet, ventilated, and closed wall boundary conditions. Again, the convergence appears to be exponential, and to be relatively insensitive to the presence of walls as well as to the wall ventilation condition. See Ref. 30 for additional discussion.

### B. Discontinuous Downwash

The problem of predicting airloads on airfoils with controls is made difficult, both computationally and theoretically, by the discontinuity in downwash at the control hinge. As is well known, a logarithmic singularity in pressure (and hence in pressure factor) arises at the hinge line so that finitely many polynomial basis functions cannot be used to give a uniform approximation to the exact solution. However, we shall show that acceptable engineering accuracy can be achieved by collocation followed by extrapolation to the limit.

To see this, consider the family of control displacement functions,

$$h_a(x) = \delta\langle x-a\rangle \qquad (42)$$

where $\delta$ is the control setting in radians and $\langle x-a\rangle$ is the unit slope ramp function corresponding to a hinge at $x=a$. For steady flow in free air, the exact value of $C_{L\delta} = 2(\cos^{-1}a + \sqrt{1-a^2})$ may be used as a basis of comparison. Table 2a lists the collocation solution convergence with $N$ for three different hinge locations: 25%, 50%, and 75% chord. Two aspects of the raw convergence are noted: 1) the convergence rate is generally $O(1/N)$, and 2) the error oscillates with a period which depends upon the hinge location. For the 50% chord hinge, the oscillation period is 2, and it is 3 for the 25%

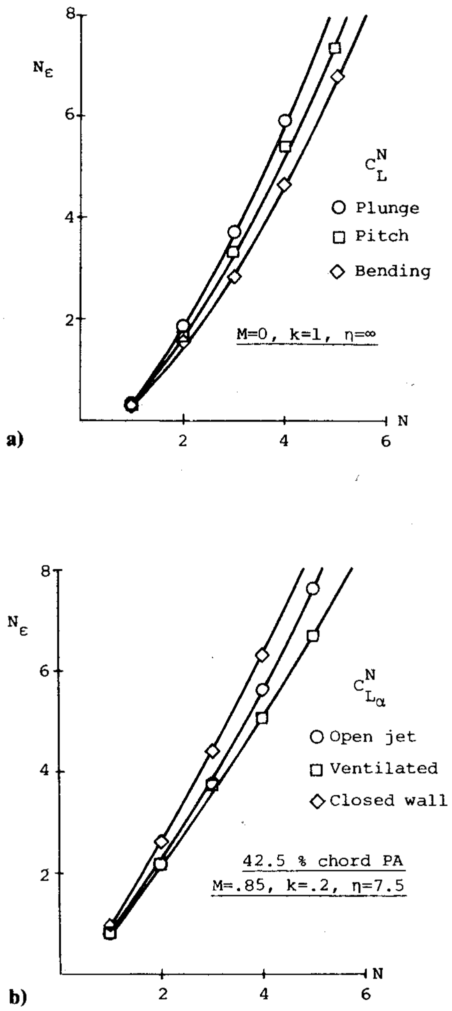


**Fig. 2 Collocation accuracy vs $N$ (smooth $w$): a) incompressible free air flow, b) compressible wind tunnel flow.**

**Table 1 Convergence of collocation (smooth $w$), $M=0$, $k=1$, $\eta=\infty$**

| $N$ | Mode | $C_L^N$ | $\lvert\epsilon\rvert$ (%)[a] | $N_\epsilon^b$ |
|---|---|---|---|---|
| 1 | Plunge | −0.90269 + 0.89375$i$ | 47.9397 | 0.32 |
| 2 | ” | −1.44015 + 1.93299$i$ | 1.3062 | 1.88 |
| 3 | ” | −1.41662 + 1.91204$i$ | 0.0181 | 3.74 |
| 4 | ” | −1.41700 + 1.91225$i$ | 0.0001 | 5.93 |
| 5 | ” | −1.41700 + 1.91225$i$ | 0.0000 | 7.67 |
| 1 | Pitch | −0.01787 + 3.59288$i$ | 57.4138 | 0.24 |
| 2 | ” | 2.72274 + 6.75414$i$ | 1.4393 | 1.84 |
| 3 | ” | 2.76154 + 6.65529$i$ | 0.0484 | 3.32 |
| 4 | ” | 2.76296 + 6.65851$i$ | 0.0039 | 5.41 |
| 5 | ” | 2.76296 + 6.65848$i$ | 0.0000 | 7.39 |
| 1 | Bending | 4.45984 + 6.30988$i$ | 59.0291 | 0.23 |
| 2 | ” | 12.22317 + 5.57437$i$ | 2.5633 | 1.59 |
| 3 | ” | 12.16652 + 5.24594$i$ | 0.1521 | 2.82 |
| 4 | ” | 12.18467 + 5.23655$i$ | 0.0022 | 4.66 |
| 5 | ” | 12.18440 + 5.23665$i$ | 0.0000 | 6.77 |

[a]Relative error defined as $\epsilon = C_L^N/C_L^{\text{exact}} - 1$. [b]Number of decimals of accuracy $N_\epsilon = \log_{10}(1/\lvert\epsilon\rvert)$.

**Table 2 Convergence of collocation (discontinuous $w$), $M=0$, $k=0$, $\eta=\infty$**

a) Raw convergence data

| | 25% chord hinge | | 50% chord hinge | | 75% chord hinge | |
|---|---|---|---|---|---|---|
| $N$ | $C_{L\delta}^N$ | $\epsilon$ (%) | $C_{L\delta}^N$ | $\epsilon$ (%) | $C_{L\delta}^N$ | $\epsilon$ (%) |
| 1 | 6.283 | +6.12 | 6.283 | +22.20 | 6.283 | +64.20 |
| 2 | 6.283 | +6.12 | 4.547 | −11.57 | 4.547 | +18.82 |
| 3 | 5.607 | −5.30 | 5.607 | + 9.06 | 3.413 | −10.82 |
| 4 | 5.957 | +0.60 | 4.803 | − 6.59 | 4.803 | +25.52 |
| 5 | 6.102 | +3.06 | 5.434 | + 5.69 | 4.129 | + 7.91 |
| 6 | 5.755 | −2.80 | 4.905 | − 4.61 | 3.595 | − 6.04 |
| 7 | 5.934 | +0.21 | 5.355 | + 4.14 | 4.429 | +15.76 |
| 8 | 6.040 | +2.02 | 4.960 | − 3.54 | 4.018 | + 5.01 |
| 9 | 5.808 | −1.90 | 5.309 | + 3.26 | 3.666 | − 4.19 |
| 10 | 5.927 | +0.11 | 4.994 | − 2.87 | 4.262 | +11.39 |
| 11 | 6.010 | +1.50 | 5.280 | + 2.69 | 3.967 | + 3.67 |
| 12 | 5.836 | −1.44 | 5.017 | − 2.42 | 3.704 | − 3.21 |

b) Convergence of periodic subsequences

| 25% chord hinge | | | 50% chord hinge | | | 75% chord hinge | | |
|---|---|---|---|---|---|---|---|---|
| $N$ | $C_{L\delta}^N$ | $N\times\epsilon$ | $N$ | $C_{L\delta}^N$ | $N\times\epsilon$ | $N$ | $C_{L\delta}^N$ | $N\times\epsilon$ |
| 3 | 5.607 | −0.159 | 2 | 4.547 | −0.232 | 3 | 3.413 | −0.325 |
| 6 | 5.755 | −0.168 | 4 | 4.803 | −0.264 | 6 | 3.595 | −0.362 |
| 9 | 5.808 | −0.171 | 6 | 4.905 | −0.276 | 9 | 3.666 | −0.377 |
| 12 | 5.836 | −0.173 | 8 | 4.960 | −0.283 | 12 | 3.704 | −0.385 |
| 15 | 5.852 | −0.174 | 10 | 4.994 | −0.287 | 15 | 3.727 | −0.390 |
| 18 | 5.864 | −0.174 | 12 | 5.017 | −0.290 | 18 | 3.743 | −0.393 |
| 21 | 5.872 | −0.174 | 14 | 5.034 | −0.292 | 21 | 3.754 | −0.395 |
| 24 | 5.878 | −0.175 | 16 | 5.047 | −0.294 | 24 | 3.763 | −0.397 |

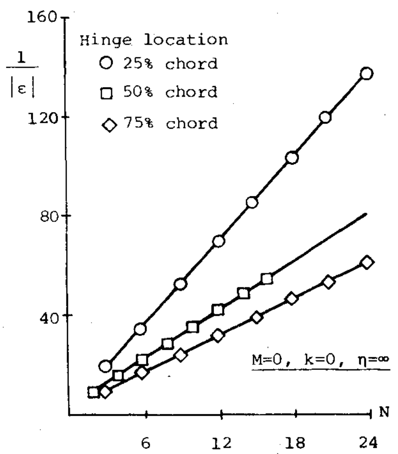


Fig. 3 Collocation accuracy in $C_{L\delta}^N$ vs $N$ (discontinuous $w$).

and 75% chord locations. Monotone convergence of appropriate subsequences is shown in Table 2b and Fig. 3.

It might thus seem, that to achieve high accuracy for airfoils with controls, a prohibitive number of terms would be required. A closer examination, however, reveals that four-decimal accuracy can be obtained with the present data, as the next section shows.

## VII. Extrapolation Method for Airfoils with Flaps

The method of extrapoaltion to the limit may be used to accelerate convergence of monotone sequences with convergence rate $O(1/n)$. This can be adapted to the problem of controls by identifying and extracting appropriate monotone subsequences.

Let $z_1, z_2, \ldots$ be a sequence of complex numbers having limit $z$,

$$\lim_{n\to\infty} z_n = z \tag{43}$$

and having $O(1/n)$ oscillatory convergence with period $p$. The error may be expressed asymptotically as

$$\epsilon_n = z - z_n = e^{2\pi i(n/p)} a_1/n + a_2/n^2 + \ldots \tag{44}$$

where $a_1, a_2, \ldots$ are the asymptotic coefficients. These properties describe the convergence behavior of the collocation method when controls are present. Such errors are strictly monotone when the number of terms equal multiples of the oscillation period times powers of 2:

$$\epsilon_p = a_1/p + a_2/p^2 + \ldots \tag{45a}$$

$$\epsilon_{2p} = a_1/2p + a_2/4p^2 + \ldots \tag{45b}$$

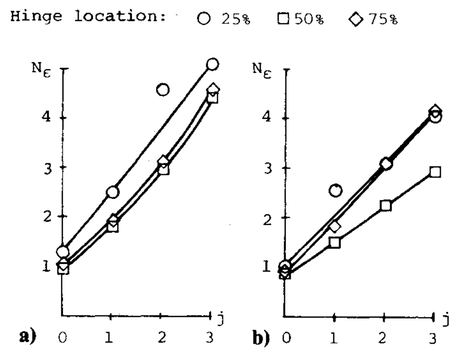


Fig. 4 Extrapolation accuracy for airfoils with controls: a) steady ($k=0$), b) unsteady ($k=1$).

Upon taking linear combinations of these monotone subsequences, one can form a new sequence

$$z_p^j = \omega_1 z_p + \omega_2 z_{2p} + \ldots + \omega_{j+1} z_{2^j p} \tag{46}$$

with index $j$ such that

$$z - z_p^j = O(1/p^{j+1}) \tag{47}$$

Then it follows that the weighting coefficients $\omega_1, \ldots, \omega_{j+1}$ must satisfy the $(j+1)\times(j+1)$ matrix equation

$$\left[\; 2^{-(m-1)(n-1)} \;\right] \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{j+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{48}$$

which always possesses a unique solution since the coefficient matrix is of Vandermonde type. For ease of discussion, we append the original sequence in the case $j=0$. Thus, the first few resulting extrapolation formulas are

$$z_p^0 = z_p \tag{49a}$$

$$z_p^1 = (-1) z_p + 2 z_{2p} \tag{49b}$$

$$z_p^2 = (1/3) z_p - 2 z_{2p} + (8/3) z_{4p} \tag{49c}$$

$$z_p^3 = (-1/21) z_p + (2/3) z_{2p} - (8/3) z_{4p} + (64/21) z_{8p} \tag{49d}$$

The effect of applying the extrapolation formulas [Eqs. (49)] to the problem of controls may be precisely assessed by considering the case of incompressible flow in free air. We

Table 3 Convergence acceleration for airfoils with controls using extrapolation, $M=0$, $k=1$, $\eta=\infty$

| | 25% chord hinge | | | | 50% chord hinge | | | | 75% chord hinge | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $j$ | $N$ | $C_{L\delta}$ | $\lvert\epsilon\rvert$, % | $N_\epsilon$ | $N$ | $C_{L\delta}$ | $\lvert\epsilon\rvert$, % | $N_\epsilon$ | $N$ | $C_{L\delta}$ | $\lvert\epsilon\rvert$, % | $N_\epsilon$ |
| 0 | 3 | 1.95154 + 4.91288$i$ | 8.390 | 1.08 | 2 | 2.10502 + 2.48091$i$ | 15.94 | 0.80 | 3 | 1.84328 + 0.73821$i$ | 13.07 | 0.88 |
| 0 | 6 | 2.04575 + 5.12878$i$ | 4.310 | 1.37 | 4 | 2.27415 + 2.66663$i$ | 9.516 | 1.02 | 6 | 1.93986 + 0.83049$i$ | 7.202 | 1.14 |
| 0 | 12 | 2.08902 + 5.24212$i$ | 2.208 | 1.66 | 8 | 2.36437 + 2.81569$i$ | 5.012 | 1.30 | 12 | 2.00082 + 0.88012$i$ | 3.741 | 1.43 |
| 0 | 24 | 2.11126 + 5.30099$i$ | 1.117 | 1.95 | 16 | 2.41334 + 2.89603$i$ | 2.580 | 1.59 | 24 | 2.03384 + 0.90545$i$ | 1.903 | 1.72 |
| 1 | 3 | 2.13995 + 5.34467$i$ | 0.303 | 2.52 | 2 | 2.44327 + 2.85236$i$ | 3.384 | 1.47 | 3 | 2.03643 + 0.92276$i$ | 1.460 | 1.84 |
| 1 | 6 | 2.13229 + 5.35547$i$ | 0.105 | 2.98 | 4 | 2.45460 + 2.96475$i$ | 0.508 | 2.29 | 6 | 2.06178 + 0.92954$i$ | 0.304 | 2.52 |
| 1 | 12 | 2.13351 + 5.35985$i$ | 0.027 | 3.57 | 8 | 2.46231 + 2.97636$i$ | 0.148 | 2.83 | 12 | 2.06686 + 0.93090$i$ | 0.072 | 3.14 |
| 2 | 3 | 2.12973 + 5.35906$i$ | 0.088 | 3.06 | 2 | 2.45837 + 3.00221$i$ | 0.563 | 2.25 | 3 | 2.07023 + 0.93180$i$ | 0.082 | 3.09 |
| 2 | 6 | 2.13392 + 5.36131$i$ | 0.007 | 4.15 | 4 | 2.46488 + 2.98023$i$ | 0.031 | 3.50 | 6 | 2.06855 + 0.93135$i$ | 0.006 | 4.19 |
| 3 | 3 | 2.13452 + 5.36163$i$ | 0.008 | 4.05 | 2 | 2.46581 + 2.97709$i$ | 0.115 | 2.94 | 3 | 2.06831 + 0.93129$i$ | 0.007 | 4.18 |

make use of the known result[31] that

$$
\begin{aligned}
C_{L_\delta} &= 2[1+ik(1-a)](\sqrt{1-a^2}+\cos^{-1}a) \\
&-[1-C(k)]\{[1+ik(2-a)]\sqrt{1-a^2}+[1+ik(1-2a)]\cos^{-1}a\} \\
&\quad -k^2[(2/3+a^2/3)\sqrt{1-a^2}-a\cos^{-1}a] \qquad (50)
\end{aligned}
$$

where $C(k)$ is the usual Theodorsen circulation function. Table 3 and Fig. 4 display the convergence characteristics of the method of extrapolation. Significant improvement results for all hinge locations in steady and unsteady flow. The fact that the number of decimals of accuracy is approximately linear with respect to the extrapolation order $j$ verifies the exponential convergence with $j$ that is predicted by Eq. (47). Further details may be found in Ref. 30.

The use of extrapolation reduces computational error by a factor of several hundred with very little (13%) increase in computer time. Whereas unextrapolated results are roughly an order of magnitude less accurate than typical wind-tunnel experimental data, the extrapolated results have computational errors an order of magnitude lower than that of experiments. Thus, by effectively eliminating computational error, interference calculations may be made with greater confidence, and a more rational comparison can be made between the mathematical model and physical tests. Two additional observations are noted. The extrapolation method is not adversely affected by compressibility or by the presence of wind tunnel walls with ventilation because the source of the computational difficulty is the inverse of the unitary part of the airfoil operator. Calculations to date bear this out. Finally, a word of caution about using extrapolation is given. It is essential that the correct subsequence be used, for otherwise, the errors will diverge. Further discussion on this point may be found in Refs. 12 and 25.

## VIII. Numerical Results from TWODI

### A. Resonance between Oscillating Airfoils and Tunnel Walls

The phenomenon of transverse acoustic resonance between an oscillating airfoil and wind-tunnel walls was discovered analytically by Runyan and Watkins[32] and later verified experimentally and computationally by Runyan et al.[33]

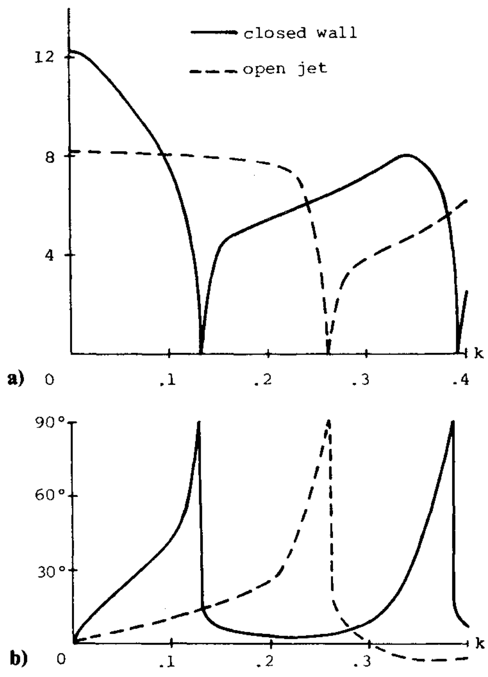


**Fig. 5 Effect of acoustic resonance on lift: a) magnitude $|C_{L_\alpha}|$, b) phase angle $\phi_{L_\alpha}$.**

Physically, whenever the oscillation frequency is such that disturbances emanating from the airfoil and reflected by the walls arrive back at the airfoil so that succeeding disturbances cancel one another, then acoustic resonance will occur. In the linearized inviscid theory, vanishingly small pressures are required near resonant frequencies to produce a given downwash; thus, airloads vanish at resonance. Runyan and Watkins[32] showed, for a closed wall tunnel, that there are an infinite number of such frequencies given by $k_n=\beta\pi(n-\frac{1}{2})/M$ for $n=1,2,...$ Bland[8,9] extended their result to ventilated tunnels governed by the slotted wall boundary condition of Eq. (7) to give

$$k_n=\beta\mu_n/M\eta \qquad (51)$$

where $\mu_n$ is given by Eq. (12d). Physically, ventilation increases resonant frequencies because it attenuates wall reflections. Equation (51) predicts this effect theoretically and indicates that the fundamental resonant frequency is doubled in going from a closed wall to an open jet.

Figure 5 shows the effect of resonance on lift for an airfoil pitching about the 42.5% chord at $M=0.85$ in a wind tunnel with height to chord ratio of 7.5. Two ventilation conditions are shown, the closed wall and the open jet. Results for intermediate ventilation conditions will lie between these two extremes. The drop in magnitude and the phase shift at resonance are seen to be quite abrupt, even more so than that originally predicted[33] for closed wall tunnels using less sophisticated computational techniques.

Figure 6 compares the pressure distribution for the closed wall case with the free air solution at $k=0.1$. Referring to Fig. 5, a significant increase in loading has been caused by the presence of walls, but it has been countered by a comparable reduction due to the onset of resonance, the latter being highly frequency sensitive. Thus, in Fig. 6a, the magnitude of pressure would seem to be similar to that for free air conditions. Figure 6b, on the other hand, indicates a marked phase shift of about 30 deg that is nearly uniform across the chord. However, at other frequencies and ventilation settings, the airloads in the tunnel can differ greatly from those of the free air condition. Clearly these effects must be given careful consideration in the planning and evaluation of unsteady tests.

### B. Combined Effect of Tunnel Ventilation and Height

Figure 7 shows the combined effect of ventilation and height to chord ratio on lift and pitching moment over the full

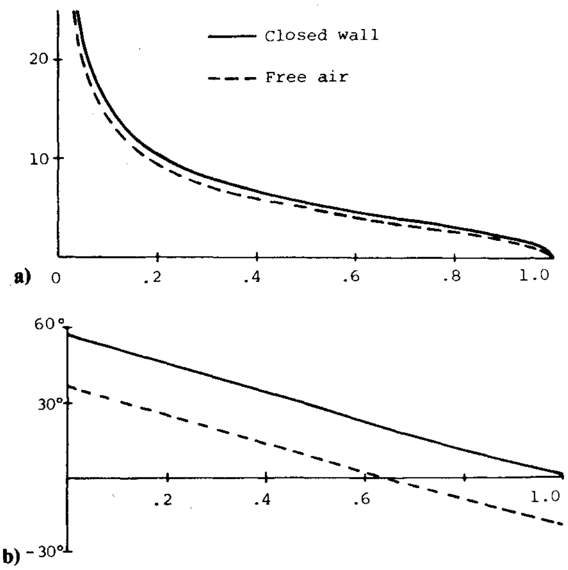


**Fig. 6 Effect of interference on pressure: a) magnitude $|\Delta p|$, b) phase angle $\phi_{\Delta p}$.**

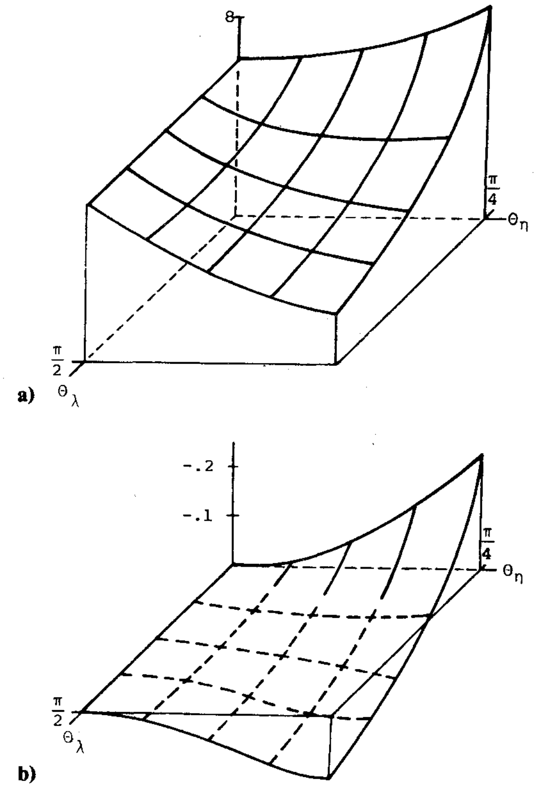


**Fig. 7 Section coefficients vs ventilation coefficient and height to chord ratio ($M=0$, $k=0$): a) lift $\partial C_L/\partial\alpha$, b) moment $\partial C_M/\partial\alpha$.**

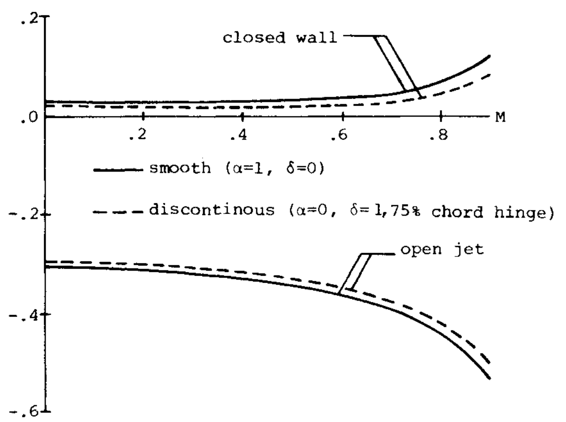


**Fig. 8 Predicted lift interference ratio vs Mach number, ventilation, and downwash type for $k=0$ and $\eta=4$.**

range of ventilation coefficient and for tunnels ranging from narrow ($\eta=1$) to free air ($\eta=\infty$). Since both $\lambda$ and $\eta$ have an infinite domain, the transformations

$$\theta_\lambda=\tan^{-1}\lambda,\quad \theta_\eta=\tan^{-1}1/\eta \tag{52}$$

have been employed to enable compact, three-dimensional graphs. The line $\theta_\eta=0$ corresponds to free air conditions and, as is to be expected, the effect of ventilation diminishes as the height to chord ratio increases.

While the trends in lift and moment are alike, their interference effects are quantitatively different. Thus, for a given height to chord ratio, the ventilation setting required to produce the same lift as in free air would not produce the free air moment, and vice versa. For flutter testing, at most one element in the aerodynamic work matrix can be adjusted to its free air value by the ventilation setting.

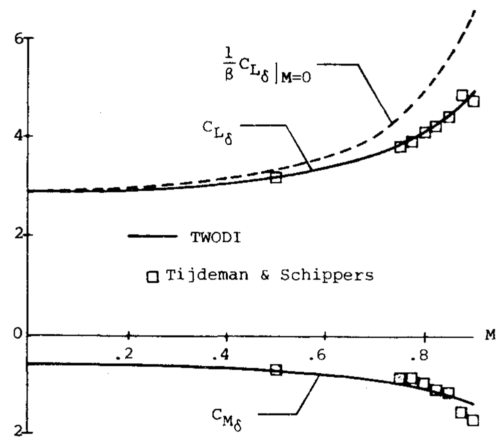


**Fig. 9 Comparison of $C_{L\delta}$ and $C_{M\delta}$ vs $M$ with experiment.**

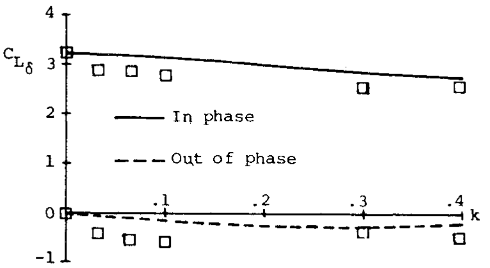


**Fig. 10 Comparison of $C_{L\delta}$ vs frequency with experiment.**

## C. Effect of Mach Number, Ventilation, and Downwash Type on Interference Ratios

Let

$$\epsilon_L=C_L/C_L^{\text{Free}}-1 \tag{53}$$

denote the lift interference ratio due to the presence of wind-tunnel walls. Figure 8 shows the effect on $\epsilon_L$ of Mach number, ventilation, and downwash type for steady flow in a wind tunnel with height to chord ratio equal to 4. The effect of Mach number increases with higher Mach numbers, showing positive interference ratios for closed wall tunnels and negative interference ratios for open jet tunnels. Also, the lift interference ratio is stronger for uniform angle of attack than for flap deflection.

## D. Comparison with Experiment

A series of steady and unsteady two-dimensional tests on airfoils with controls has been performed at high subsonic and transonic speeds by Tijdeman and associates[34-36] in the NLR pilot tunnel. This is a closed circuit tunnel capable of producing sonic speeds in a rectangular test section 420 mm wide by 550 mm high. Ventilation is provided by longitudinal slots with an open area ratio of 0.1.

A NACA 64A006 airfoil with 180 mm chord and 45 mm trailing-edge control was tested at various Mach numbers and frequencies. Figure 9 shows a comparison between predicted and measured values of lift and moment in steady flow. The agreement is excellent. For additional comparison, the value of $C_{L\delta}$ at $M=0$, multiplied by the compressibility factor $1/\beta$, is also shown. This indicates that the rate of increase in airloads with respect to Mach number is less in ventilated tunnels than in free air. For these calculations, the ventilation coefficient, $\lambda=2.014$, was selected to match the experimental value of lift at $M=0.5$. Airloads were computed using second-order extrapolation.

Figure 10 compares predicted and measured values of lift vs frequency at $M=0.5$. While lift is matched at $k=0$ through the choice of ventilation coefficient as described earlier, the comparison at nonzero frequencies is definitely not as good. Although experimental difficulties were initially encountered[34] in obtaining correct transfer functions for the tube-transducer measurement system, this problem is believed to have been corrected[35] (see Ref. 36 for further discussion), and hence should not affect the comparison. However, a closer examination of test reports[35] reveals that the mean control deflection was not quite zero for the unsteady conditions. This will influence unsteady airloads through variable coefficients of the unsteady nonlinear PDE field equations and through differential ventilation between upper and lower tunnel walls. This could account for at least part of the discrepancy, especially since nonlinear effects become significant at relatively lower Mach numbers in unsteady flow. Also, we suspect on physical grounds that the slotted wall boundary condition of Eq. (7) is not fully satisfactory for general unsteady flow.

## IX. Conclusions

In this paper, we have discussed the computation of two-dimensional airloads acting on oscillating airfoils with leading- and trailing-edge controls in ventilated subsonic wind tunnels. This was accomplished by solving Bland's integral equation by an efficient collocation method using orthogonal airfoil polynomial pairs. These polynomials provide an elegant reformulation of the Söhngen inversion formula in the case of steady flow in free air where generalized Fourier coefficients of pressure factor are identically equal to generalized Fourier coefficients of downwash. Since nonresonant unsteady flow in a compressible, ventilated wind tunnel is a smooth parametric extension of this fundamental special case, a powerful computational procedure emerges for the more general case.

Extremely efficient solutions with exponential convergence rate result for smooth downwashes; typically, only five terms in the pressure series suffice to produce six-decimal computational accuracy for $k=O(1)$. Downwashes due to controls yield a slower $O(1/N)$ rate of convergence, which has been significantly accelerated through extrapolation to the limit.

Using TWODI, a variety of computations have been made and several physical effects studied. The combined effect of ventilation and height to chord ratio have been calculated over the full range of interest of these parameters. For steady flow, increasing ventilation always decreases airloads, and increasing the height to chord ratio may increase or may decrease airloads, depending upon the value of the ventilation coefficient and upon the airload quantity being computed. Lift interference effects have been shown to be downwash dependent at various Mach numbers in closed wall and open jet tunnels. Weaker interference ratios are found for control displacements than for uniform angle-of-attack displacements. The effect of transverse acoustic resonance between an oscillating model and the tunnel walls has been discussed. Significant changes in magnitude and phase angle can be seen, and the effect of ventilation depends strongly upon frequency and its relation to resonance. Limited comparisons with experimental data indicate excellent agreement for lift and moment on thin symmetrical airloads (NACA 64A006) with controls (75% chord hinge) in steady flow over a wide range of subsonic Mach numbers ($M=0.5$-$0.9$) in relatively narrow, ventilated tunnels. Good agreement with experiment is obtained for unsteady flow.

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